Systems of Equations

Introduction –

A *system* of ODE's is just a set of more than one ODE in some number of unknown functions $x_1, x_2, x_3, \ldots$ where $x_i = x_i(t)$ for all i.

Ex: $x_1'' + 3x_1x_2 - tx_1 = 0$

$x_2' + 7t^2x_1^2x_2 = 0$

where $x_1 = x_1(t)$, $x_2 = x_2(t)$ is a system of 2 ODE's in the 2 unknown functions x_1 and x_2.

There are 2 reasons we care about systems –

1) Systems occur quite naturally in applications.
2) Higher order equations can always be reduced to systems of lower order equations, in particular 1st order equations which may be easier to solve.

For example, recall that if $\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$ is the familiar position vector of a particle as a function of time and the particle moves in a force field $\vec{F} = F_x\hat{i} + F_y\hat{j} + F_z\hat{k}$ where F_x, F_y, F_z are constants then trivially

$$m\ddot{x} = F_x$$
$$m\ddot{y} = F_y$$
$$m\ddot{z} = F_z$$

More generally though it may be that

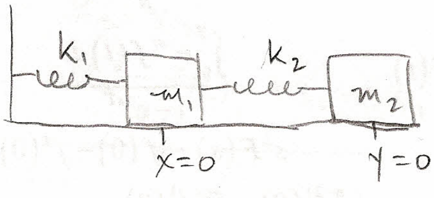
$$F_i = F_i(t, x, y, z, \dot{x}, \dot{y}, \dot{z}) \quad , i = 1,2,3$$

which leads to the system

$$m\ddot{x} = F_x(t, x, y, z, \dot{x}, \dot{y}, \dot{z})$$
$$m\ddot{y} = F_y(t, x, y, z, \dot{x}, \dot{y}, \dot{z})$$
$$m\ddot{z} = F_z(t, x, y, z, \dot{x}, \dot{y}, \dot{z})$$

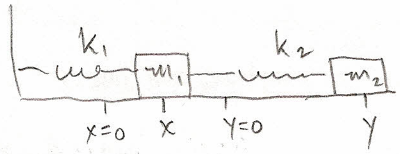
which is, in general, very difficult (or impossible) to solve exactly.

As a second example consider the coupled oscillator (without friction)



Now we have two masses, m_1 and m_2, and two springs with spring constants k_1 and k_2. $x=0$ and $y=0$ are the equilibrium positions of m_1 and m_2 respectively.

we can now displace the two masses –



and note the following –

1) The displacement of spring 1 from equilibrium is x.
2) The displacement of spring 2 is $y - x$.
3) Spring 1 acts on m_1 in a direction opposite displacement.
4) Spring 2 acts on m_2 in a direction opposite displacement.
5) Spring 2 acts on m_1 in the <u>same</u> direction as displacement.

Then applying Newton's 2^{nd} law, $F = ma = m\ddot{x}$ to each mass we have –

$$m_1\ddot{x} = -k_1 x + k_2(y - x)$$
$$m_2\ddot{y} = -k_2(y - x)$$

which is a system of two 2^{nd} order equations in x and y.

More generally, a system of m particles interacting in n-dimensions will result in a system of $m \times n$ 2^{nd} order equations.

The second use of systems is the reduction of higher order equations. Any higher order equation or system of higher order equations can <u>always</u> be reduced to a system of 1^{st} order equations which (hopefully) is easier to solve.

<u>Ex</u>: The second order equation

$$ay'' + by' + cy = f(t), \quad y = y(t)$$

can be transformed to a system of two 1^{st} order equations by making the substitutions

$$x_1 = y$$
$$x_2 = x_1' = y'$$

Then the original equation becomes

$$a x_2' + b x_2 + c x_1 = f(t)$$

and we have a system of two first order equations –

$$x_1' = x_2$$
$$a x_2' = f(t) - b x_2 - c x_1$$

Similarly, a 3rd order equation can be transformed into a system of three 1st order equations -

Ex: $x''' - tx'' + xx' - x^2 = \cos t$ where $x = x(t)$

Let $x_1 = x$
$x_2 = x_1' = x'$
$x_3 = x_2' = x_1'' = x''$

Then the 3rd order equation becomes the system of 1st order equations

$$x_1' = x_2$$
$$x_2' = x_3$$
$$x_3' - tx_3 + x_1 x_2 - x_1^2 = \cos t$$

or -

$$\boxed{\begin{aligned} x_1' &= x_2 \\ x_2' &= x_3 \\ x_3' &= tx_3 - x_1 x_2 + x_1^2 + \cos t \end{aligned}}$$

In general, a system of n 1st order equations is of the form -

$$x_1' = F_1(t, x_1, x_2, \ldots, x_n)$$
$$x_2' = F_2(t, x_1, x_2, \ldots, x_n)$$
$$\vdots$$
$$x_n' = F_n(t, x_1, x_2, \ldots, x_n)$$

and generally can't be solved exactly.

If the system is linear (all equations are linear) then it is of the form -

$$x_1' = a_{11}(t) x_1 + a_{12}(t) x_2 + \ldots + a_{1n}(t) x_n + f_1(t)$$
$$x_2' = a_{21}(t) x_1 + a_{22}(t) x_2 + \ldots + a_{2n}(t) x_n + f_2(t)$$
$$\vdots$$
$$x_n' = a_{n1}(t) x_1 + a_{n2}(t) x_2 + \ldots + a_{nn}(t) x_n + f_n(t)$$

If $f_1(t) = f_2(t) = \ldots = f_n(t) = 0$ then the system is <u>homogeneous</u>. Otherwise it is <u>non-homogeneous</u>.

For arbitrary a_{ij} this is still usually impossible to solve but we can always solve the system if a_{ij} = constant for all i, j and the system is homogeneous -

$$x_1' = a_{11} x_1 + a_{12} x_2 + \ldots + a_{1n} x_n$$
$$x_2' = a_{21} x_1 + a_{22} x_2 + \ldots + a_{2n} x_n$$
$$\vdots$$
$$x_n' = a_{n1} x_1 + a_{n2} x_2 + \ldots + a_{nn} x_n$$

Method of elimination –

We can solve a linear, homogeneous of 1st order equations with constant coefficients by a method similar to that used for systems of linear algebraic equations –

Ex! Solve $x' = 4x - 3y$ ①

$y' = 6x - 7y$ ②

subject to the initial conditions –

$x(0) = 2, \; y(0) = -1$

We eliminate one variable by solving one equation & substituting into the other – Solving ② for x we find –

$$x = \tfrac{1}{6}y' + \tfrac{7}{6}y$$

note we also need x' –

$$x' = \tfrac{1}{6}y'' + \tfrac{7}{6}y'$$

Substituting into ① –

$$\tfrac{1}{6}y'' + \tfrac{7}{6}y' = 4\left(\tfrac{1}{6}y' + \tfrac{7}{6}y\right) - 3y$$

This is a second order equation in y –

$$y'' + 3y' - 10 = 0$$

We know how to solve this – if $y = e^{kt}$ then

$$k^2 + 3k - 10 = 0$$

$$(k+5)(k-2) = 0$$

$$k = -5, 2$$

and $y = c_1e^{2t} + c_2e^{-5t}$

we have x from above –

$$x = \tfrac{1}{6}y' + \tfrac{7}{6}y$$

$$x = \tfrac{1}{6}\{2c_1e^{2t} - 5c_2e^{-5t}\} + \tfrac{7}{6}\{c_1e^{2t} + c_2e^{-5t}\}$$

$$x = \tfrac{3}{2}c_1e^{2t} + \tfrac{1}{3}c_2e^{-5t}$$

applying the IC's we find

$$x(0) = \tfrac{3}{2}c_1 + \tfrac{1}{3}c_2 = 2$$

$$y(0) = c_1 + c_2 = -1$$

Solving these two equations you find $c_1 = 2, c_2 = -3$ and

$$\left.\begin{aligned} x &= 3e^{2t} - e^{-5t} \\ y &= 2e^{2t} - 3e^{-5t} \end{aligned}\right\} \text{ solves the IVP.}$$

That's okay for 2 equations but it gets ridiculous for larger systems. There is a better way, the matrix method.

The Phase Plane

A system of two 1^{st} order equations has a graphic representation similar to a direction field for a single 1^{st} order equation.

Ex: $x' = -y$
$y' = x$

We can solve this system.

$$\left.\begin{aligned} y &= -x' \\ y' &= -x'' \end{aligned}\right\} \text{ where } x = x(t),\ y = y(t)$$

Substituting into second equation we have

$$-x'' = x$$
$$x'' + x = 0$$

and $x = A\cos t + B\sin t$

Since $y = -x'$ we find

$$y = A\sin t - B\cos t$$

So $\left.\begin{aligned} x &= A\cos t + B\sin t \\ y &= A\sin t - B\cos t \end{aligned}\right\}$ is a set of parametric equations for $x(t)$ and $y(t)$.

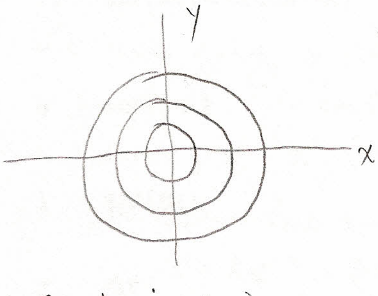
We can eliminate t from this set of equations –

$$x^2+y^2 = A^2\cos^2 t + B^2\sin^2 t + 2AB\sin t\cos t + A^2\sin^2 t + B^2\cos^2 t - 2AB\sin t\cos t$$

or $x^2+y^2 = A^2+B^2$

or $x^2+y^2 = r^2$ where $r^2 = A^2+B^2$

So the solutions $(x(t), y(t))$ lie on a circle where the radius of the circle is determined by the initial conditions, i.e. for a given set of IC's, (x,y) lie on one circle for all t.

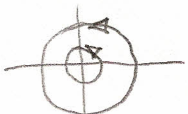


For example, if $x(0) = 1,\ y(0) = 0$ then $x^2+y^2 = 1$

if $x(0) = 1,\ y(0) = 1$ then $x^2+y^2 = 2$

etc...

Each circle is a trajectory, there is exactly one trajectory for a given set of IC's.
The set of all trajectories is called the phase plane.
The solutions are called trajectories because they describe the motion of a particle in the x-y plane which obeys the set of 1^{st} order ODE's.
The only thing we don't know is the direction of travel. Is it clockwise

 or is it counterclockwise  ?

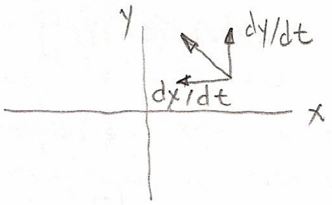
To answer this question we look at the derivatives.

If $x > 0, y > 0$ (1st quadrant) then

$$x' = -y < 0$$

and $y' = x > 0$

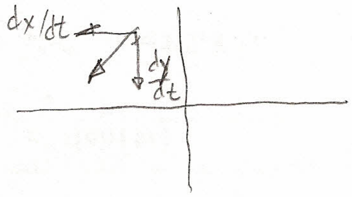
So the motion is in the direction of $-x$ and positive y –



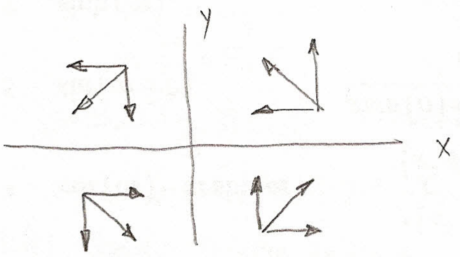
In the second quadrant $x < 0$ and $y > 0$.

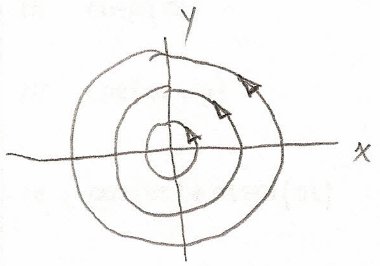
Then $x' = -y < 0$

and $y' = x < 0$



you can repeat this for each quadrant if you want but the result is that the motion is counter-clockwise –





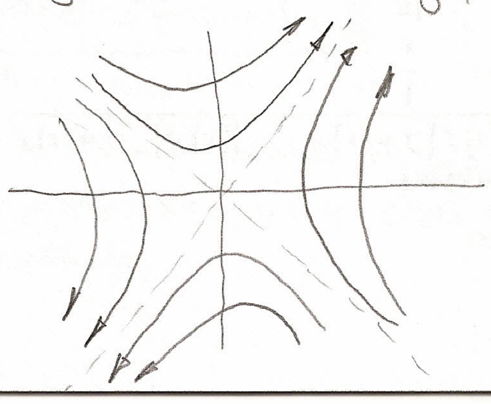
It was easy to find the trajectories in this problem. In general you will want to use software such as p-plane.

For example, if $x' = y$ and $y' = 2x + y$ then $y' = x''$ and $x'' - x' - 2x = 0$.

You can solve this and find

$$x = Ae^{-t} + Be^{2t}$$
$$y = -Ae^{-t} + 2Be^{2t}$$

These trajectories are hard to graph by hand but using p-plane –



you should find a pair of asymptotes and a fixed point at (0,0). Do it!!